Power law violation of the area law in quantum spin chains

Ramis Movassagh and Peter W. Shor

Northeastern / M.I.T.

QIP, Sydney, Jan. 2015

Quantifying entanglement: SVD

(Schmidt Decomposition, aka SVD) Suppose $|\psi_{AB}\rangle$ is the pure state of a composite system, AB. Then there exists orthonormal states $|\Phi_A^\alpha\rangle$ for A and orthonormal states $|\theta_B^\alpha\rangle$ for B

$$|\psi_{AB}\rangle = \sum_\alpha \lambda_\alpha |\Phi_A^\alpha\rangle \otimes |\theta_B^\alpha\rangle$$

where λ_α's satisfy $\sum_\alpha |\lambda_\alpha|^2 = 1$ known as **Schmidt numbers.**

The number of non-zero Schmidt numbers is called the **Schmidt rank, χ,** of the state (a quantification of entanglement) .

Quantifying entanglement: Entropy

Another measure of entanglement is **entanglement entropy**.

Recall we had

$$|\psi_{AB}\rangle = \sum_{\alpha} \lambda_\alpha |\Phi_A^\alpha\rangle \otimes |\theta_B^\alpha\rangle$$

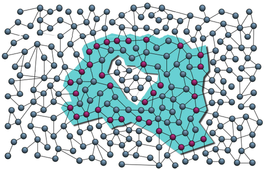
where λ_α's satisfy $\sum_\alpha |\lambda_\alpha|^2 = 1$. The entanglement entropy is:

$$S \equiv -\sum_{\alpha} |\lambda_\alpha|^2 \log |\lambda_\alpha|^2$$

where $|\lambda_\alpha|^2 = p_\alpha$ are the probabilities.

Area laws

Area Laws



Picture from Eisert, Cramer, Plenio, *Rev. Mod. Phys.* **82** (2010)

Area law: Suppose you have a Hamiltonian with only local interactions, and a quantum system is in the ground state of the Hamiltonian. Then the entropy of entanglement between two subsystems of a quantum system is proportional to the area of the boundary between them.

Area law and implications for simulability

1D gapped local systems obey an area law.
[M.B. Hastings (2007)]
This makes them easy to simulate on a classical comuter.

- Matrix Product States, DMRG, PEP, etc. work very well for 1D systems with an area law.

higher dimensions

It is believed that higher-dimensional gapped systems obey an area law (open).
For critical systems, it is believed the area law contains an extra log factor.

In D spatial dimensions one expects:

$$\begin{aligned} S &\sim L^{D-1} &: \quad &\text{Gapped} \\ S &\sim L^{D-1}\log(L) &: \quad &\text{Critical} \end{aligned}$$

Phase transitions

For 1-dimensional spin chains at critical points, the continuous limit is generally a conformal field theory:

- Entropy of entanglement: $O(\log n)$,
- Spectral gap: $O(1/n)$.

Basic idea that started our research

- Simulating 1D spin chains with local Hamiltonians is BQP-complete. (Gottesman, Irani).
- 1D spin chains with low entanglement are classically simulable.

Therefore: there must be 1D spin chains with high entanglement.

arxiv:1001.1006

Movassagh, Farhi, Goldstone, Nagaj, Osborne, Shor (2010)

We investigated spin chains with qudits of dimension d, interaction is a projection dimension r.

- The ground state is frustration-free but entangled when $d \leq r \leq d^2/4$.
- we could compute the Schmidt ranks,
- We could not obtain definitive results on the spectral gap or the entanglement entropy.

arXiv:0901.1107

Irani (2010)

There are Hamiltonians whose ground states have:

- spectral gap $O(1/n^c)$,
- entanglement entropy $O(n)$,
- complicated Hamiltonians,
- high-dimensional spins.

Bravyi et al 2012

Bravyi, Caha, Movassagh, Nagaj, Shor (2012)

There are Hamiltonians whose ground states

- have spectral gap $O(1/n^c)$,
- have entanglement entropy $O(\log n)$,
- are frustration free,
- have spins of dimension 3.

New result

Movassagh, Shor (2014)

There are Hamiltonians whose ground states

- have spectral gap $O(1/n^c)$, $c \geq 2$
- have entanglement entropy $O(\sqrt{n})$,
- are frustration free,
- have spins of dimension $2s+1$, s > 1.

Another new result

Movassagh, Shor (2014)

There are Hamiltonians whose ground states

- numerically have spectral gap $O(1/n^c)$, $c \geq 2$
- have entanglement entropy $O(\sqrt{n})$,
- are unique,
- are not frustration free,
- have spins of dimension $2s+1$, s $>$ 1.

These properties do not depend on the boundary conditions.

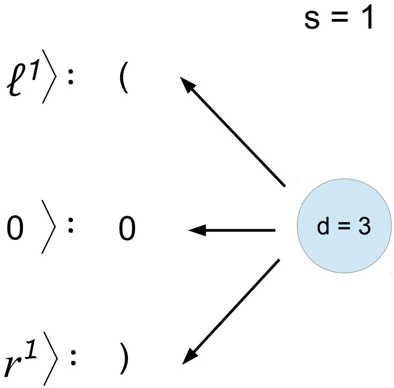
Summary of the new result

- The '**Motzkin state**', $|\mathscr{M}_{2n,s}\rangle$ is the *unique ground state* of the local Hamiltonian
- **Entanglement entropy** violates the area law:

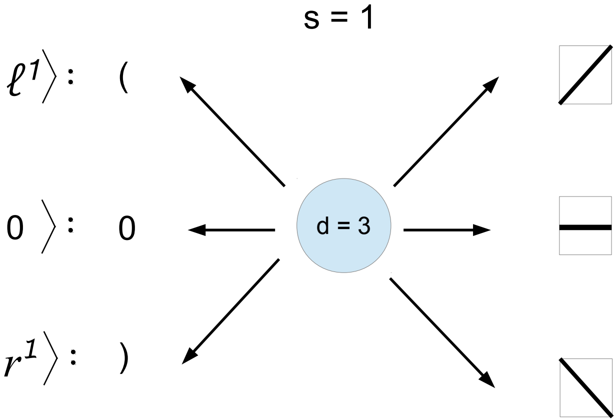
$$S(n) = c_1(s)\log_2(s)\sqrt{n} + \frac{1}{2}\log(n) + c_2(s)$$
$$\chi = \frac{s^{n+1}-1}{s-1}.$$

- The gap **upper bound:** $\mathscr{O}\left(n^{-2}\right)$. Brownian excursion and universality of Brownian motion.
- The gap **lower bound:** $\Omega((n^{-c})$, $c \gg 1$. Fractional matching and statistics of random walks

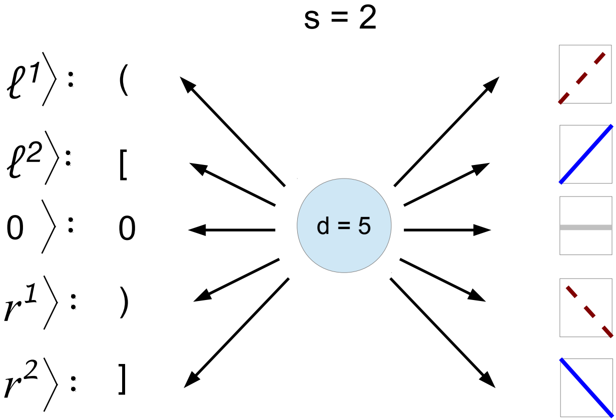
States: $d = 2s + 1$



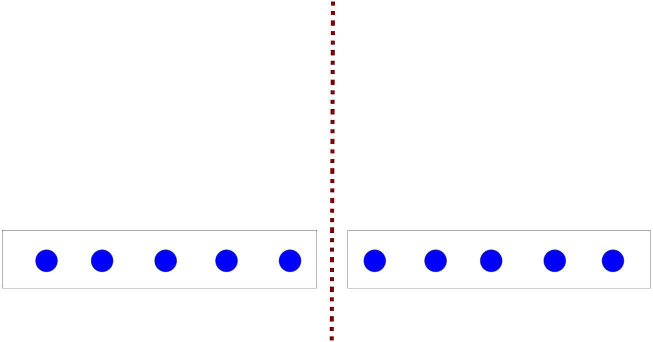
States



$s \geq 1$

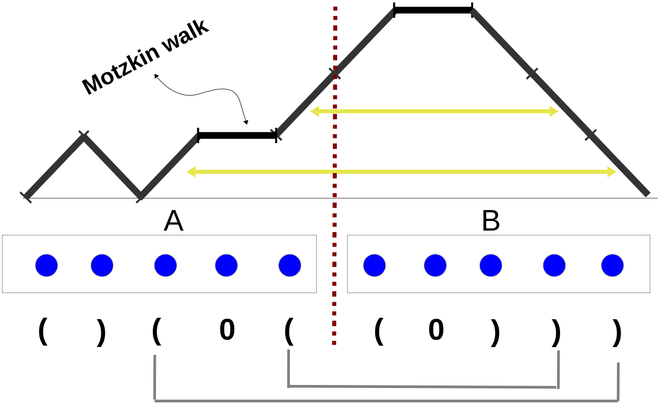


Ground states



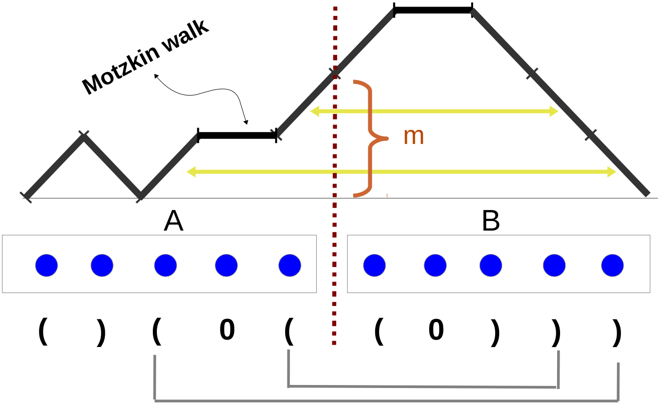
How to quantify entanglement

Entanglement of **Motzkin States** is due to the mutual information between halves



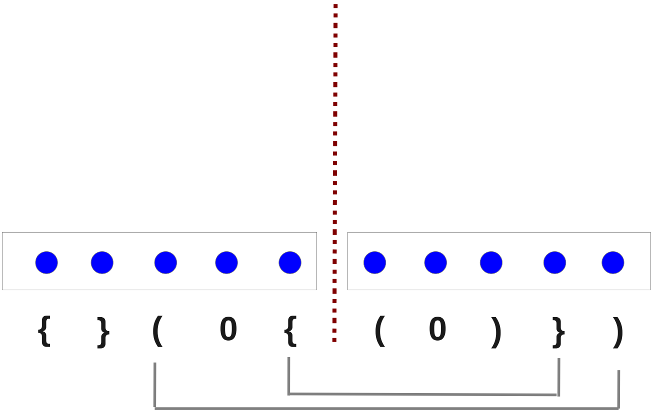
How to quantify entanglement

Entanglement of **Motzkin States** is due to the mutual information between halves



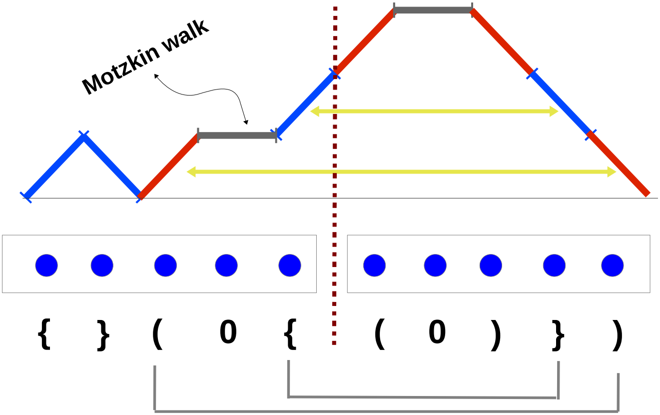
More than one type of 'parenthesis' e.g., $s = 2$

Suppose there are two types **(** and **{** to match



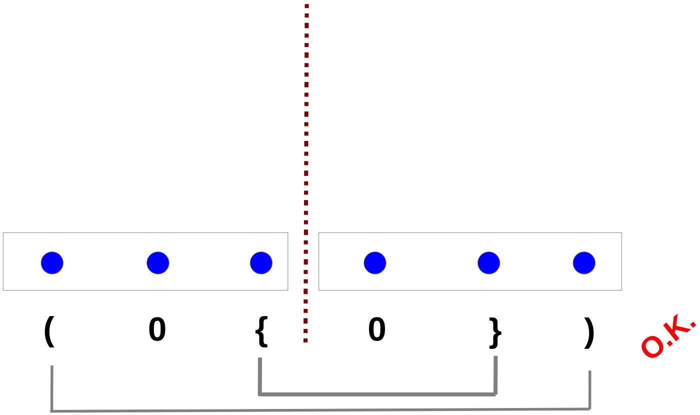
More than one type of 'parenthesis' e.g., $s = 2$

Entanglement of **Colored Motzkin States** is due to the mutual information between halves



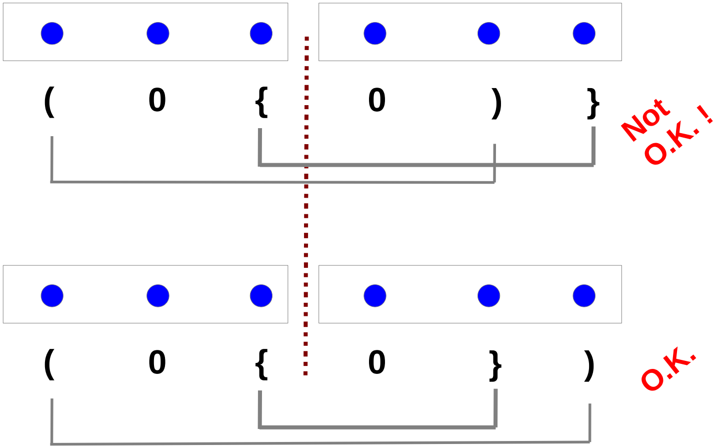
Subtlety for $s > 1$

Suppose there are two types **(** and **{** to match



Matching that does NOT work

Suppose there are two types **(** and **{** to match



The ground state $s = 1$

$$|\mathscr{M}_{2n,s}\rangle = \frac{1}{\sqrt{M_{2n}}} \sum_p |p^{\text{th}} \text{ Motzkin walk}\rangle$$

e.g., $2n = \{2, 4\}$

$$\begin{aligned} |\mathscr{M}_2\rangle &= \frac{1}{2}\{\ |00\rangle + |\ell r\rangle\} \\ |\mathscr{M}_4\rangle &= \frac{1}{9}\{\ |0000\rangle + |00\ell r\rangle + |0\ell 0r\rangle + |\ell 00r\rangle + |0\ell 0r\rangle \\ &\quad + |\ell 0r0\rangle + |\ell r00\rangle + |\ell r\ell r\rangle + |\ell\ell rr\rangle\} \end{aligned}$$

The ground state $s \geq 1$

$$|\mathscr{M}_{2n,s}\rangle = \frac{1}{\sqrt{M_{2n}}} \sum_p |p^{\text{th}} \text{ s-colored Motzkin walk}\rangle$$

e.g., $2n = 2$

$$|\mathscr{M}_{2,s}\rangle \sim \left\{ |00\rangle + \sum_{k=1}^{s} |\ell^k r^k\rangle \right\}$$

Entanglement: Schmidt rank χ

$$p_{n,m,s} = \frac{M^2_{n,m,s}}{N_{n,s}} , \qquad N_{n,s} \equiv \sum_{m=0}^{n} s^m M^2_{n,m,s}, \tag{1}$$

Geometric sum on m gives $\chi = \frac{s^{n+1}-1}{s-1}$ and entanglement entropy is

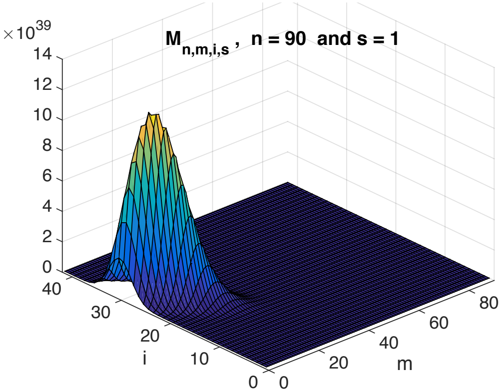
$$S(\{p_{n,m,s}\}) = -\sum_{m=0}^{n} s^m p_{n,m,s} \log_2 p_{n,m,s}. \tag{2}$$

Combinatorial factors

$M_{n,m,s}$: number of walks starting at zero ending at height m with s total colors

$$
\begin{aligned}
M_{n,m,s} &= \frac{m+1}{n+1}\sum_{i\geq 0}\binom{n+1}{i+m+1,\; i,\; n-2i-m} s^{i} \\
&\equiv \sum_{i\geq 0} M_{n,m,i,s}
\end{aligned}
$$

Location of the Saddle



Saddle point: $\frac{M_{n,m,s,i+1}}{M_{n,m,s,i}} = 1$, $\frac{M_{n,m+1,s,i}}{M_{n,m,s,i}} = 1$.

Saddle point approximation

Turning the sum into an integral (carefully) and performing saddle point integration $m = \alpha\sqrt{n}$:

$$
\begin{aligned}
M_{n,m,s} &\approx \frac{1}{2\sqrt{\pi\sigma}\, n^{3/2}} \left(\frac{\sqrt{s}}{\sigma}\right)^{n+1} \alpha s^{-\alpha\sqrt{n}/2} \exp\left(-\frac{\alpha^2}{4\sigma}\right) \;. \\
S &\approx 2\log(s)\, \sqrt{\frac{2\sigma}{\pi}}\, \sqrt{n} + \frac{1}{2}\log n + \gamma - \frac{1}{2} + \frac{1}{2}\log(2\pi\sigma) \;, \\
\sigma &\equiv \frac{\sqrt{s}}{1+2\sqrt{s}} \;.
\end{aligned}
$$

Underlying Hamiltonian 'implements' local moves

$$H = \left\{ \sum_{k=1}^{s} r_k\rangle_1\langle r_k + \sum_{k=1}^{s} \ell_k\rangle_{2n}\langle \ell_k \right\} + \sum_{j=1}^{2n-1} \Pi_{j,j+1},$$

$\Pi_{j,j+1}$ projects onto the span of $(\forall\ k, = 1, \cdots, s)$

$$\begin{aligned}
\frac{1}{\sqrt{2}}\left[|0\ell^k\rangle - |\ell^k 0\rangle\right] \quad &: \quad 0\ell^k \longleftrightarrow \ell^k 0 \\
\frac{1}{\sqrt{2}}\left[|0r^k\rangle - |r^k 0\rangle\right] \quad &: \quad 0r^k \longleftrightarrow r^k 0 \\
\frac{1}{\sqrt{2}}\left[|00\rangle - |\ell^k r^k\rangle\right] \quad &: \quad 00 \longleftrightarrow \ell^k r^k \\
\Pi^{cross} \quad &= \quad \sum_{k\neq i} \ell_k r_i\rangle\langle r_i \ell_k .
\end{aligned}$$

Meaning of terms in Hamiltonian

The terms

$$\frac{1}{\sqrt{2}}\left[|0\ell^k\rangle - |\ell^k 0\rangle\right] \quad : \quad 0\ell^k \longleftrightarrow \ell^k 0$$
$$\frac{1}{\sqrt{2}}\left[|0r^k\rangle - |r^k 0\rangle\right] \quad : \quad 0r^k \longleftrightarrow r^k 0$$
$$\frac{1}{\sqrt{2}}\left[|00\rangle - |\ell^k r^k\rangle\right] \quad : \quad 00 \longleftrightarrow \ell^k r^k$$

enforce an *equal superposition* of all walks which can be reached by

- switching ℓ^k and 0
- switching r^k and 0,
- replacing $\ell^k r^k$ by 00.

Meaning of terms in Hamiltonian

The cross terms

$$\Pi^{cross} = \sum_{k \neq i} \ell_k r_i \rangle \langle r_i \ell_k.$$

ensure that the types of parentheses match.

The boundary terms

$$\left\{ \sum_{k=1}^{s} r_k \rangle_1 \langle r_k + \sum_{k=1}^{s} \ell_k \rangle_{2n} \langle \ell_k \right\}$$

ensure that the walk is balanced.

Gap Upper-Bound

We show that the gap is $\mathscr{O}(n^{-2})$.

Gap: Upper bound I

We want any state $|\phi\rangle$ such that

$$\langle\phi|H|\phi\rangle = \mathcal{O}\left(n^{-2}\right), \quad \langle\phi_{\text{ground}}|H|\phi\rangle < \frac{1}{2}.$$

Then

$$|\phi\rangle = \alpha_g|\phi_g\rangle + \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \dots$$

and

$$\begin{aligned}\langle\phi|H|\phi\rangle &= \alpha_1^2\langle\phi_1|H|\phi_1\rangle + \alpha_2^2\langle\phi_2|H|\phi_2\rangle + \alpha_3^2\langle\phi_3|H|\phi_3\rangle + \dots \\ &\geq (1-\alpha_g^2)\langle\phi_1|H|\phi_1\rangle\end{aligned}$$

Gap: Upper bound II

$$\phi\rangle = \frac{1}{\sqrt{M_{2n}}} \sum_p e^{\left\{2\pi i\theta \left(\text{Area of } {}_p\text{th Motzkin walk}\right)\right\}} |p^{\text{th}} \text{ Motzkin walk}\rangle$$

$$\langle \mathscr{M}_{2n} | \phi \rangle = \frac{1}{M_{2n}} \sum_p e^{\left\{2\pi i\theta \left(\text{Area of } {}_p\text{th Motzkin walk}\right)\right\}}$$

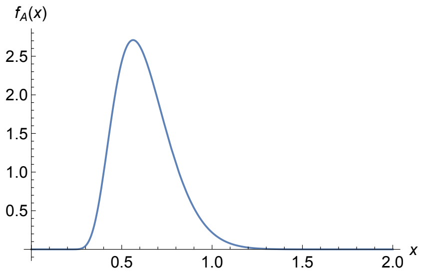
Gap: Upper bound II

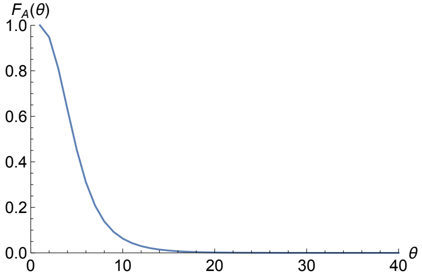
$$\phi\rangle = \frac{1}{\sqrt{M_{2n}}} \sum_p e^{\left\{2\pi i\theta \left(\text{Area of } p^{\text{th}} \text{ Motzkin walk}\right)\right\}} |p^{\text{th}} \text{ Motzkin walk}\rangle$$

$$\langle \mathscr{M}_{2n}|\phi\rangle = \frac{1}{M_{2n}} \sum_p e^{\left\{2\pi i\theta \left(\text{Area of } p^{\text{th}} \text{ Motzkin walk}\right)\right\}}$$

$$\lim_{n\to\infty} \langle \mathscr{M}_{2n}|\phi\rangle \approx F_A(\theta) \equiv \int_0^\infty f_A(x)\, e^{2\pi i x\theta} dx \quad .$$

Gap: Upper bound III





$$f_A(x) = \frac{2\sqrt{6}}{x^2} \sum_{j=1}^{\infty} v_j^{2/3} \; e^{-v_j} \; U\left(-\frac{5}{6}, \frac{4}{3}; v_j\right) \qquad x \in [0, \infty)$$

$v_j = 2|a_j|^3/27x^2$ where a_j are the zeros of the Airy function.

Gap Lower-Bound

$\Theta(n^{-c})$, for some constant c.
We use the same techniques as in Bravyi, Caha, Movassagh, Nagaj, Shor.

- the projection lemma relating Motzkin walks and Dyck walks,
- proving rapid mixing with the canonical paths method,
- fractional matchings.

This Hamiltonian isn't completely satisfactory

- requires boundary conditions to have unique ground state.

Without the boundary conditions, there would be $\binom{n+2}{2}$ ground states, each coming from a superposition of unbalanced walks:

$$(\; 0 \; (\; (\;) \; 0 \;) \; ()$$

How can we eliminate these ground states without boundary conditions?

We add an energy penalty for ℓ and r — i.e., for '(' and ')'.

Hamiltonian with extermal magnetic field.

How can we prove anything about these states?
The argument that the gap is at most $\mathcal{O}(n^{-2})$ still holds.
Only have to worry about the gap in two cases:

- Unbalanced walks
- Superposition of balanced walks with positive coefficients.

Hamiltonian with extermal magnetic field.

The gap for unbalanced walks:

Let ε be the energy penalty for ℓ^k, r^k.
We can use perturbation theory (backed up with numerics) to show that these ground states have an increased energy of around $c\varepsilon^2/n$.

Hamiltonian with extermal magnetic field.

The gap for states in the balanced subspace.

There is a polynomial gap in the balanced subspace in the Hamiltonian without an energy penalty. It appears from numerics (on chains of relatively small length) that the gap in this case is Numerics seem to show that the gap in the balanced subspace with an energy penalty is also $\Theta(n^{-2})$.

Open problems

- Is there a continuum limit for these Hamiltonians?
- Can we rigorously prove the results with an external magnetic field?
- Are there frustration-free Hamiltonians with unique ground states which violate the area law by large factors?

Lastly...

Thank you